Unsupervised Image Segmentation with Spatial Triplet Markov Trees

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Outline

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- **Dense models** → approximating methods → model very complex phenomena (e.g. artifacts on medical images)
- Combined with deep learning → many **top current results** in image processing (e.g. Variational Autoencoders)

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- Segmentation criteria: $\rightarrow$ Maximum A Posteriori (MAP)
  $\rightarrow$ Maximum Posterior Mode (MPM)

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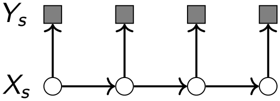
  - $\boldsymbol{X}$ is a Markovian process and $p(\boldsymbol{y}|\boldsymbol{x}) = \prod_{s \in \mathcal{S}} p(y_s|x_s)$

Hidden Markov Models: classical models

Hidden Markov Chain (HMC) (Baum, Petrie, et al. 1970)



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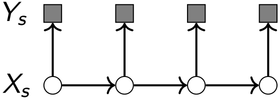

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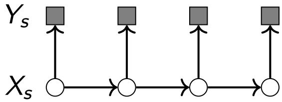

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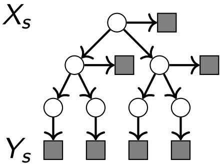

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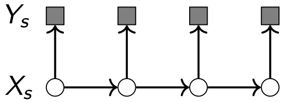

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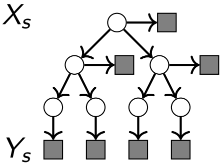

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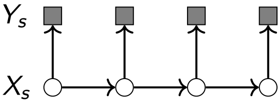

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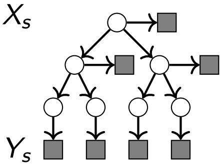

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inference $\rightarrow$ **direct, exact computations** with Forward-Backward based algorithms

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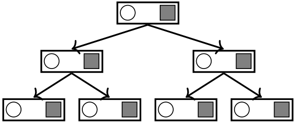
- Triplet models integrate **auxiliary random variables** $\rightarrow$ link with deep learning models

Pairwise and triplet assumptions

Pairwise Markov Tree (Pieczynski 2002)

X_s, Y_s



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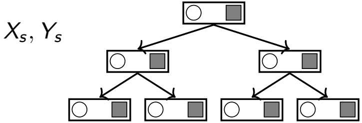

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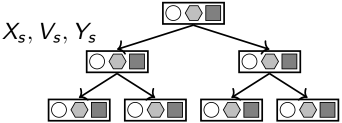
Triplet Markov Tree (TMT) (Courbot et al. 2018)



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- Neither $p(\boldsymbol{x})$, $p(\boldsymbol{y})$, $p(\boldsymbol{v})$, $p(\boldsymbol{x}, \boldsymbol{y})$, $p(\boldsymbol{y}, \boldsymbol{v})$, nor $p(\boldsymbol{x}, \boldsymbol{v})$ are necessarily Markovian distributions

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We now present the Spatial Triplet Markov Tree model:

$\rightarrow$ It takes advantage of the increased modeling possibilities
$\rightarrow$ It enhances the spatial correlations between random variables

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Spatial Triplet Markov Trees (STMTs)

Distribution of STMTs (Gangloff et al. 2020) **(this paper)**:

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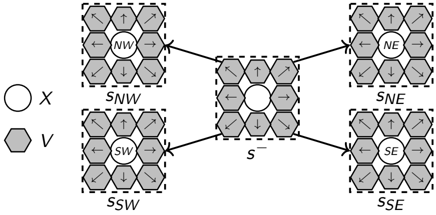
- Quadtrees: each site s^- has four sons $(s^{NW}, s^{NE}, s^{SE}, s^{SW})$ (except for last layer):



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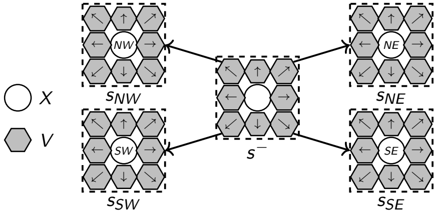
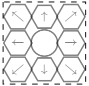


- We consider only observations Y_s at the finer resolution.

Designing the auxiliary process in STMTs



Propagation of spatial information: the same color indicates the same probability law

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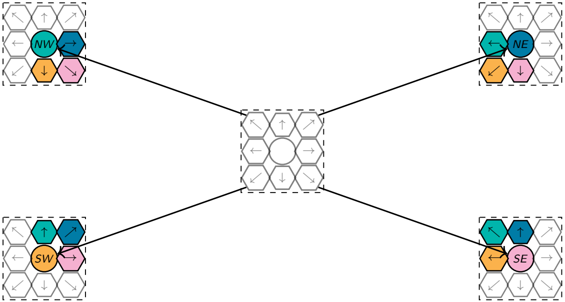

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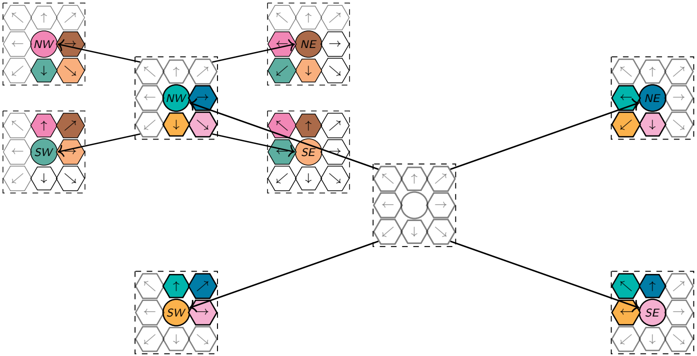

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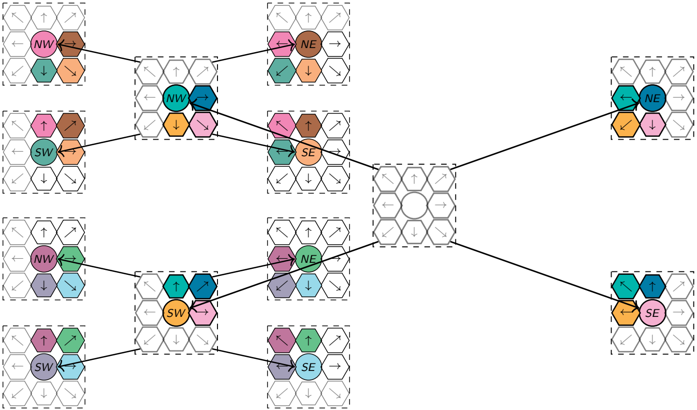

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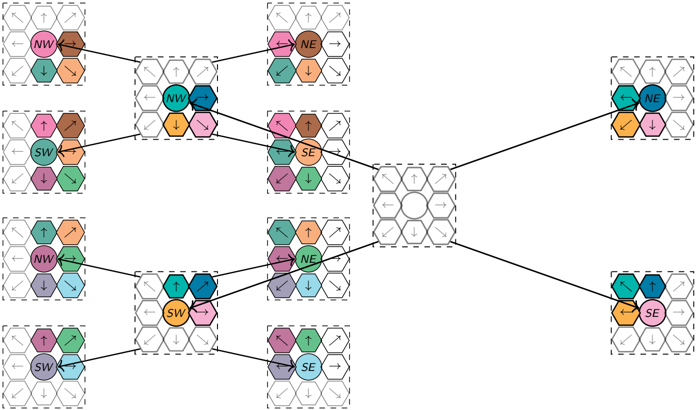

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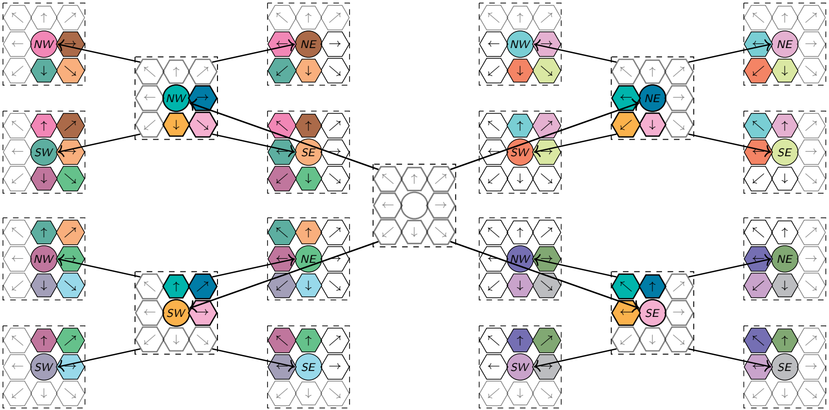

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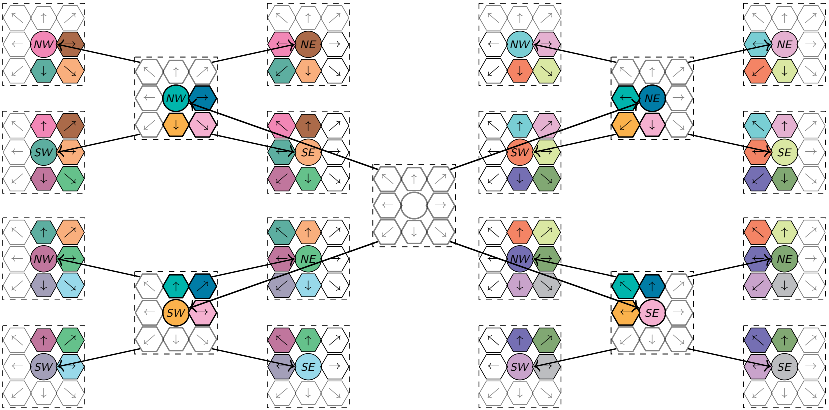

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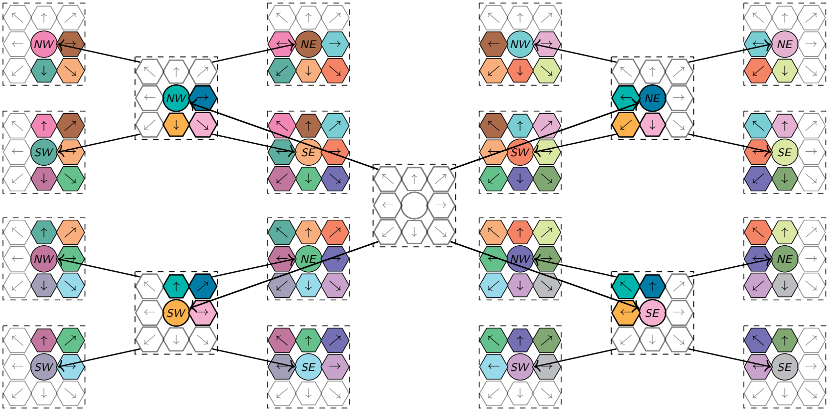

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STMTs: numerical applications

- Ising-like transitions to propagate spatial homogeneity:

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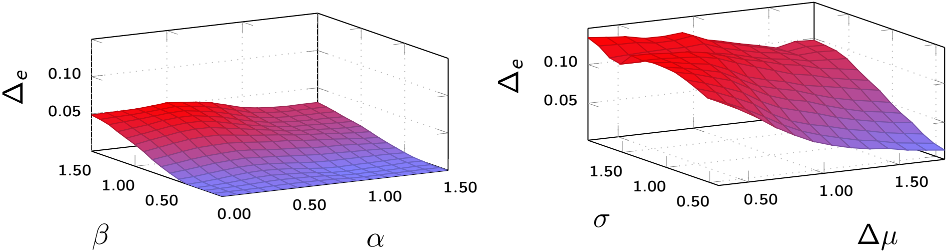
$$p(x_s|x_{s^-}, \boldsymbol{v}_{s^-}) = \frac{1}{Z}\exp\left(\alpha\delta_{x_s}^{x_{s^-}} + \sum_{v_{s^-}\in\boldsymbol{v}_{s^-}} \beta\delta_{x_s}^{v_{s^-}}\right), \text{ with } (\alpha,\beta)\in\mathbb{R}_+^2.$$

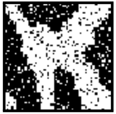
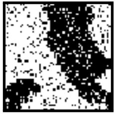
- Simulate STMTs and restore with STMTs and HMTs (known α, β parameters)



With $\Delta_e = err_{HMT} - err_{STMT}$: relative error rate

$\rightarrow$ For all model parameters and noise levels STMT performs better than HMT

Clamped sampling experiment

MF (Potts potential)		...		Average difference rate to MRF (100 simulations)
STMT (Potts-like potential)		...		6.8%
MT (Potts-like potential)		...		14.2%

Clamped samplings of the original $\boldsymbol{X}$ *process (last layer only)*

→ STMTs seem to capture the best the spatial context created by Markov Fields (MFs)

Outline

Unsupervised parameter estimation

Case of independent Gaussian observations

We introduce Linear Least Square (LLS) estimator for α^t and β^t

$$[\alpha^t, \beta^t] = (B^T B)^{-1} B^T A$$

where the generic term of vector A is, $\forall s \in \mathcal{S}^L$, $\forall (x_s, x_s') \in \Omega^2$,

$$A_s = \ln \frac{p(x_s, x_{s^-}, \boldsymbol{v}_{s^-}|\boldsymbol{y})}{p(x_s', x_{s^-}, \boldsymbol{v}_{s^-}|\boldsymbol{y})} - \ln \frac{1}{\sqrt{2\pi\sigma_{x_s}^2}} + \frac{(y_s - \mu_{x_s})^2}{2\sigma_{x_s}^2} + \ln \frac{1}{\sqrt{2\pi\sigma_{x_s'}^2}} + \frac{(y_s - \mu_{x_s'})^2}{2\sigma_{x_s}^2}$$

and the generic line of matrix B is

$$B_{s,:} = \left[\delta_{x_s}^{x_{s^-}} - \delta_{x_s'}^{x_{s^-}}, \sum_{v_{s^-} \in \boldsymbol{v}_{s^-}} \left(\delta_{x_s}^{v_{s^-}} - \delta_{x_s'}^{v_{s^-}} \right) \right]$$

Unsupervised parameter estimation

Case of independent Gaussian observations

Maximum Likelihood (ML) estimator for $\boldsymbol{\mu}^t$ and $\boldsymbol{\sigma}^t$, $\forall \omega \in \Omega$:

$$\mu_\omega^t = \frac{1}{\sum_{s\in\mathcal{S}^L} \mathbb{1}_{\{x_s^t=\omega\}}} \sum_{s\in\mathcal{S}^L} y_s \mathbb{1}_{\{x_s^t=\omega\}}$$

and

$$\sigma_\omega^t = \left(\frac{1}{\sum_{s\in\mathcal{S}^L} \mathbb{1}_{\{x_s^t=\omega\}}} \sum_{s\in\mathcal{S}^L} \left(y_s - \mu_\omega^t\right)^2 \mathbb{1}_{\{x_s^t=\omega\}} \right)^{\frac{1}{2}}$$

Unsupervised parameter estimation

Algorithm 1: Iterative Parameter Estimation for Trees for STMTs.
Case $\Omega = \{\omega_1, \omega_2\}$

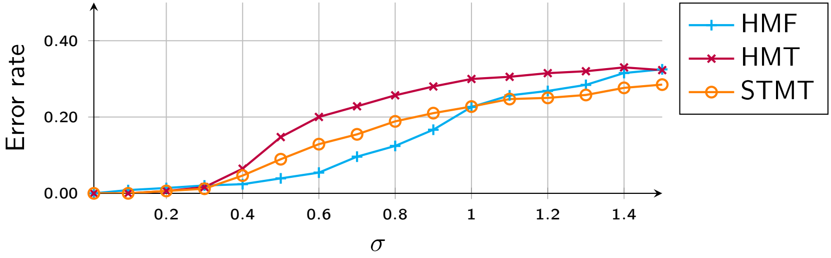
Data: $\boldsymbol{\theta}^0 = \{\alpha^0, \beta^0, \mu_0^0, \mu_1^0, \sigma_0^0, \sigma_1^0\}$, an initial set of parameters,
$\boldsymbol{y}$, the observations.

Result: $\boldsymbol{\theta}^* = \{\alpha^*, \beta^*, \mu_0^*, \mu_1^*, \sigma_0^*, \sigma_1^*\}$, the estimated parameters.

```
t ← 1
while convergence is not attained do
    1. MPM estimation:
    x̂_s^{MPM,t} = argmax_{x_s} p(x_s | y, θ^{t−1}), ∀s ∈ S
    2. Estimation with the complete data (x̂^{MPM,t}, y):
    • LLS estimator for α^t and β^t
    • ML estimator for μ_0^t and μ_1^t
    • ML estimator for σ_0^t and σ_1^t
    θ^t ←= {α^t, β^t, μ_0^t, μ_1^t, σ_0^t, σ_1^t}
    t ← t + 1
```

Unsupervised image segmentation experiment

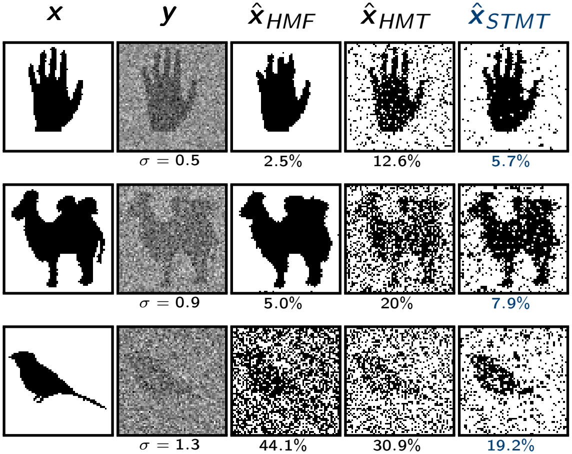
Comparing Hidden Markov Fields (HMFs), HMTs and STMTs in unsupervised segmentation:



Error rate in unsupervised segmentation function of the noise level

→ STMTs greatly improve HMT results
→ STMTs closer to HMF results

Unsupervised image segmentation experiment



Unsupervised segmentation HMF, HMT and STMT models

Outline

Conclusion

Summary

- STMTs are generalizations of HMTs
  $\rightarrow$ increased modeling possibilities
  $\rightarrow$ strengthened spatial correlations
- Inference remains exact and deterministic

Conclusion

Summary

- STMTs are generalizations of HMTs
  $\rightarrow$ increased modeling possibilities
  $\rightarrow$ strengthened spatial correlations
- Inference remains exact and deterministic

Perspectives

- *Spatial* correlations induced $\rightarrow$ theoretical links between STMTs and HMFs ?
- STMTs as the variational distribution for variational inference in trees with semi-cycles as in (Gangloff et al. 2020) for 2D segmentation

References I

[1] L. E. Baum and T. Petrie. "Statistical inference for probabilistic functions of finite state Markov chains". In: *The annals of mathematical statistics* 37.6 (1966), pp. 1554–1563.

[2] L. E. Baum, T. Petrie, G. Soules, and N. Weiss. "A maximization technique occurring in the statistical analysis of probabilistic functions of Markov chains". In: *The annals of mathematical statistics* 41.1 (1970), pp. 164–171.

[3] C. M. Bishop. *Pattern recognition and machine learning*. Springer, 2006.

[4] J.-B. Courbot, E. Monfrini, V. Mazet, and C. Collet. "Triplet markov trees for image segmentation". In: *SSP 2018: IEEE Workshop on Statistical Signal Processing*. IEEE Computer Society, 2018, pp. 233–237.

[5] H. Gangloff, J.-B. Courbot, E. Monfrini, and C. Collet. "Spatial Triplet Markov Trees for auxiliary variational inference in Spatial Bayes Networks". In: *Stochastic Modeling Techniques and Data Analysis international conference (SMTDA'20)*. 2020. In press.

[6] S. Geman and D. Geman. "Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images". In: *IEEE Transactions on pattern analysis and machine intelligence* 6 (1984), pp. 721–741.

[7] I. Goodfellow, Y. Bengio, and A. Courville. *Deep learning*. MIT Press, 2016.

[8] I. Gorynin, H. Gangloff, E. Monfrini, and W. Pieczynski. "Assessing the segmentation performance of pairwise and triplet Markov models". In: *Signal Processing* 145 (2018), pp. 183–192.

[9] D. Koller and N. Friedman. *Probabilistic graphical models: principles and techniques*. MIT Press, 2009.

[10] J.-M. Laferté, P. Pérez, and F. Heitz. "Discrete Markov image modeling and inference on the quadtree". In: *IEEE Transactions on image processing* 9.3 (2000), pp. 390–404.

[11] K. P. Murphy. *Machine learning: a probabilistic perspective*. MIT Press, 2012.

[12] W. Pieczynski. "Arbres de Markov couple". In: *Comptes Rendus Mathématique* 335.1 (2002), pp. 79–82.